# The Mathematics Of Stock Option Valuation - Part Seven
# Put Option Payoff Distribution

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In this white paper we will develop the mathematics to calculate the probability distribution (mean and variance) of end-of-term put option payoffs. To that end we will work through the following hypothetical problem...

## Our Hypothetical Problem

Assume that we purchased a put option on the common stock of ABC Company and want to estimate the probability distribution of end-of-term option payoffs. We are given the following model parameters...

**Table 1: Model Parameters**

| Symbol | Description | Value |
|---|---|---|
| $A_0$ | Share price at time zero | \$30.00 |
| $X_T$ | Put option exercise price at time $T$ | \$25.00 |
| $P_0$ | Black-Scholes put option value at time zero | \$3.70 |
| $\kappa$ | Expected continuous-time return mean | 0.1133 |
| $\sigma$ | Expected continuous-time return volatility | 0.3000 |
| $\phi$ | Expected continuous-time dividend yield | 0.0296 |
| $\alpha$ | Continuous-time risk-free rate | 0.0407 |
| $T$ | Term in years | 5.0000 |

Answer the following questions...

**Question 1**: What is the mean and variance of random put option payoffs at time $T$?

**Question 2**: Graph the ratio of option price to the present value of expected option payoff at time $T$.

**Question 3**: What is the probability that the option payoff will be greater than \$10.00 per share?

## Generic Asset Price Equations

We will define the variable $\kappa$ to the continuous-time asset return mean, the variable $\phi$ to be the continuous-time dividend yield, the variable $\sigma$ to be asset return volatility, and the variable $T$ to be time in years. The equations for asset return mean and variance over the time interval $[0,T]$ are...

$$m = \left(\kappa - \phi - \frac{1}{2}\,\sigma^2\right) T \;\text{ ...and... }\; v = \sigma^2\,T \tag{1}$$

We will define the variable $A_T$ to be random asset price at time $T$ and the variable $\theta$ to be random asset return over the time interval $[0,T]$. Using Equation (1) above, the equation for random asset price at time $T$ as a function of asset price at time zero is...

$$A_T = A_0\,\text{Exp}\left\{\theta\right\} \;\text{ ...where... }\; \theta \sim N\left[m, v\right] \tag{2}$$

Using Equations (1) and (2) above, the equation for the probability density function for normally-distributed random asset returns is... [2]

$$\text{PDF} = \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2}\,\frac{(\theta - m)^2}{v}\right\}\delta\theta \tag{3}$$

We will define the variable $P_T$ to be the payoff on a put option on the asset at time $T$ and the variable $X_T$ to be the exercise price of that put option. Using Equation (2) above, the equation for option payoff at time $T$ is...

$$P_T = \text{Max}\Big[X_T - A_T, 0\Big] \tag{4}$$

We will define the variable $a$ to be the asset return at which the option is at-the-money. Using Equations (2) and (4) above, the equation for the asset return where the option is at-the-money is...

$$\text{if... } A_0 \,\text{Exp}\Big\{a\Big\} \text{ ...then... } a = \ln\left(\frac{X_T}{A_0}\right) \tag{5}$$

Using the equations above, the equation for expected option payoff at time $T$ is...

$$\text{Expected payoff} = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2}\frac{(\theta - m)^2}{v}\right\}\Big(X_T - A_T\Big)\delta\theta \tag{6}$$

We will define the function CNDF(x,n,v) to be the cumulative normal distribution function where the variable $x$ is a normally-distributed random variable with mean $m$ and variance $v$. This function tells us the probability that the random return $\theta$ drawn from a normal distribution with mean $m$ and variance $v$ will be less than some threshold value $x$. The equation for this function is...

$$\text{CNDF}(x, m, v) = \text{Prob}\Big[\theta < x\Big] = \int\limits_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\}\delta\theta \tag{7}$$

The function in Excel that is the equivalent of Equation (7) above is...

$$\text{Excel function CNDF}(x, m, v) = \text{NORMDIST}(x, m, \sqrt{v}, true) \tag{8}$$

## Payoff Distribution - First Moment

Using Equations (2) and (6) above, the equation for the first moment of the distribution of random put option payoffs is...

$$\text{FM} = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2}\frac{(\theta - m)^2}{v}\right\}\left(X_T - A_0\,\text{Exp}\Big\{\theta\Big\}\right)\delta\theta \tag{9}$$

We will make the following integral definitions...

$$I_1 = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2}\frac{(\theta - m)^2}{v}\right\}\delta\theta$$
$$I_2 = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2}\frac{(\theta - m)^2}{v}\right\}\text{Exp}\Big\{\theta\Big\}\,\delta\theta \tag{10}$$

Using the integral definitions in Equation (10) above, we can rewrite Equation (9) above as...

$$\text{FM} = X_T \times I_1 - A_0 \times I_2 \tag{11}$$

Using Equation (7) above, the solutions to the integrals in Equation (11) above are... [1]

$$I_1 = \text{CNDF}(a, m, v)$$
$$I_2 = \text{Exp}\left\{m + \frac{1}{2}\,v\right\}\text{CNDF}(a, m + v, v) \tag{12}$$

Using Equation (12) above, the solution to Equation (11) above is...

$$\text{FM} = X_T\,\text{CNDF}(a, m, v) - A_0\,\text{Exp}\left\{m + \frac{1}{2}\,v\right\}\text{CNDF}(a, m + v, v) \tag{13}$$

## Payoff Distribution - Second Moment

Using Equations (2) and (6) above, the equation for the second moment of the distribution of put random option payoffs is...

$$\mathrm{SM} = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(\theta-m)^2}{v}\right\}\left[X_T - A_0\,\mathrm{Exp}\left\{\theta\right\}\right]^2 \delta\theta \tag{14}$$

Note that we can rewrite Equation (14) above as...

$$\mathrm{SM} = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(\theta-m)^2}{v}\right\}\left[X_T^2 - 2\,X_T A_0\,\mathrm{Exp}\left\{\theta\right\} + A_0^2\,\mathrm{Exp}\left\{2\,\theta\right\}\right]\delta\theta \tag{15}$$

We will make the following integral definition...

$$I_3 = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(\theta-m)^2}{v}\right\}\mathrm{Exp}\left\{2\,\theta\right\}\delta\theta \tag{16}$$

Using the integral definitions in Equations (10) and (16) above, we can rewrite Equation (15) above as...

$$\mathrm{SM} = X_T^2 \times I_1 - 2\,X_T A_0 \times I_2 + A_0^2 \times I_3 \tag{17}$$

Using Equation (7) above, the solution to the integral in Equation (16) above is... [1]

$$I_3 = \mathrm{Exp}\left\{2\,(m+v)\right\}\mathrm{CNDF}(a, m+2\,v, v) \tag{18}$$

Using Equations (12) and (18) above, the solution to Equation (17) above is...

$$\mathrm{SM} = X_T^2\,\mathrm{CNDF}(a,m,v) - 2\,X_T A_0\,\mathrm{Exp}\left\{m + \frac{1}{2}\,v\right\}\mathrm{CNDF}(a, m+v, v) + A_0^2\,\mathrm{Exp}\left\{2\,(m+v)\right\}\mathrm{CNDF}(a, m+2\,v, v) \tag{19}$$

## Payoff Distribution - Mean and Variance

Using the equations above, the equation for the mean of the distribution of random option payoffs is...

$$\mathrm{Mean} = \text{First moment} \;\; \text{...see...} \;\; \text{Equation (13) above} \tag{20}$$

Using the equations above, the equation for the variance of the distribution of random option payoffs is...

$$\mathrm{Variance} = \text{Second moment} - \left[\text{First moment}\right]^2 \;\; \text{...see...} \;\; \text{Equations (13) and (19) above} \tag{21}$$

Note that the distribution of random option payoffs is lognormal.

We need to develop the mathematics to calculate the probability that option payoff will be greater than some threshold value. If we define the threshold value to be $V$, the asset return over the time interval $[0,T]$ where option payof equals our threshold value is...

$$\text{if...} \;\; V = X_T - A_0\,\mathrm{Exp}\left\{b\right\} \;\; \text{...then...} \;\; b = \ln\left(\frac{X_T - V}{A_0}\right) \tag{22}$$

Using Equations (1), (7) and (22) above, the probability that option payoff will be greater than some threshold value $V$ is...

$$\mathrm{Prob}\left[\text{Option payoff } \geq V\right] = \mathrm{CNDF}(b, m, v) \tag{23}$$

## The Answer to Our Hypothetical Problem

Using Equation (1) above and the data in Table 1 above, the equations for return mean and variance are...

$$m = \left(0.1133 - 0.0296 - \frac{1}{2} \times 0.30^2\right) \times 5 = 0.1938 \text{ ...and... } v = .30^2 \times 5 = 0.4500 \tag{24}$$

Using Equation (5) above and the data in Table 1 above, the equation for the at-the-money return is...

$$a = \ln\left(\frac{25.00}{30.00}\right) = -0.1823 \tag{25}$$

Using Equations (12), (18) and (24) above, the integral values are...

$$\begin{aligned} I_1 &= \text{CNDF}(-0.1823, 0.1938, 0.4500) = 0.2875 \\ I_2 &= \text{Exp}\left\{0.1938 + \frac{1}{2} \times 0.4500\right\} \text{CNDF}(-0.1823, 0.1938 + 0.4500, 0.4500) = 0.1658 \\ I_3 &= \text{Exp}\left\{2 \times (0.1938 + 0.4500)\right\} \text{CNDF}(-0.1823, 0.1938 + 2 \times 0.4500, 0.4500) = 0.1035 \end{aligned} \tag{26}$$

Using Equations (11) and (26) above and the data in Table 1 above, the equation for the first moment of the distribution is...

$$\text{FM} = 25.00 \times 0.2875 - 30.00 \times 0.1658 = 2.21 \tag{27}$$

Using Equations (17) and (26) above and the data in Table 1 above, the equation for the second moment of the distribution is...

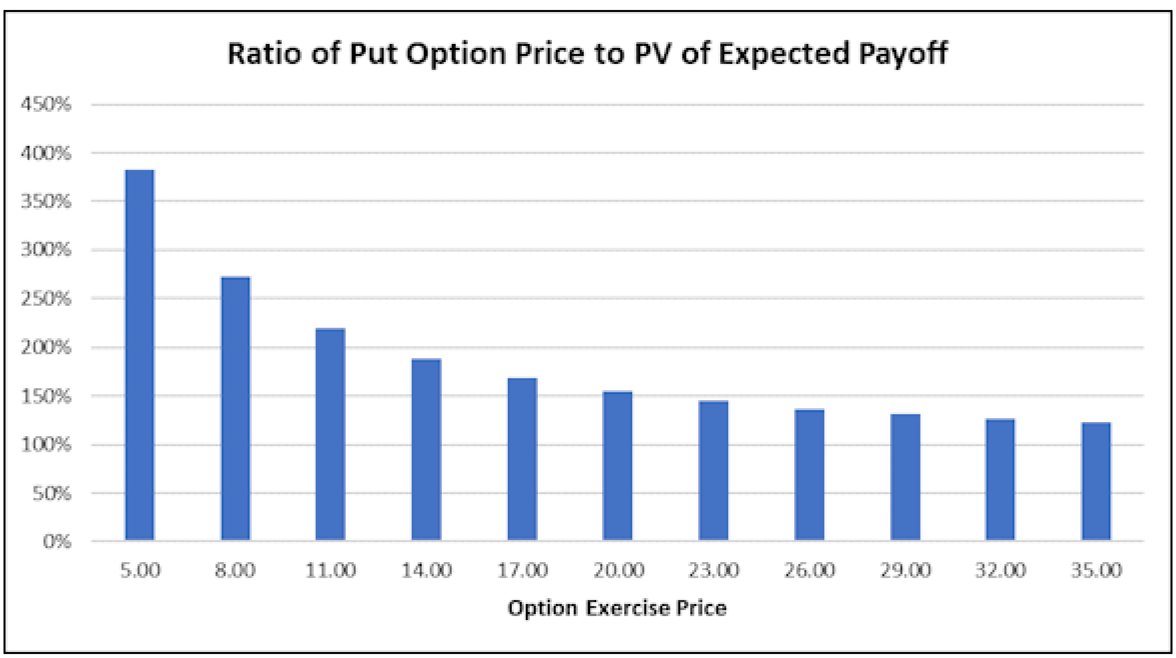
$$\text{SM} = 25.00^2 \times 0.2875 - 2 \times 25.00 \times 30.00 \times 0.1658 + 30^2 \times 0.1035 = 24.16 \tag{28}$$

**Question 1**: What is the mean and variance of random option payoffs at time $T$?

Using Equations (20), (21), (27), and (28) above, the mean and variance of the answer to the question is...

$$\text{mean} = \text{FM} = 2.21 \text{ ...and... } \text{variance} = 24.16 - 2.21^2 = 19.26 \tag{29}$$

**Question 2**: Graph the ratio of option price to the present value of expected option payoff at time $T$.




Notes: Option price is derived from the Black-Scholes equation and the present value of expected option payoff at time $T$ via Equation (13) above discounted at the risk-free rate.

**Question 3**: What is the probability that the option payoff will be greater than \$10.00 per share?

Using Equation (22) above, the equation for the threshold return is...

$$b = \ln\left(\frac{25.00 - 10.00}{30.00}\right) = -0.6931 \tag{30}$$

Using Equation (23) and (30) above, the answer to the question is...

$$\text{Prob}\Bigg[\text{Option payoff } \geq 10.00\Bigg] = \text{CNDF}(-0.6931, 0.1938 + 0.4500, 0.4500) = 0.0930 \tag{31}$$

# References


[1] Gary Schurman, *Option Valuation Integral Solutions - Integral One,* July, 2021.

[2] Gary Schurman, *The Calculus of the Normal Distribution,* October, 2010.